

## SOME NEW DEVELOPMENTS IN SMALL AREA ESTIMATION

J.N.K. Rao<sup>1</sup>

### ABSTRACT

Small area estimation has received a lot of attention in recent years due to growing demand for reliable small area statistics. Traditional area-specific estimators may not provide adequate precision because sample sizes in small areas are seldom large enough. This makes it necessary to employ indirect estimators based on linking models. Basic area level and unit level models have been extensively studied in the literature to derive empirical best linear unbiased prediction (EBLUP), empirical Bayes (EB) and hierarchical Bayes (HB) small area estimators and associated measures of variability. In this paper, I will cover several important new developments related to model-based small area estimation.

KEY WORDS: Area Level Model; Jackknife; MSE Estimation; Unit Level Model.

### RÉSUMÉ

L'estimation sur de petits domaines a suscité beaucoup d'attention ces dernières années dû à la demande croissante de statistiques fiables sur ces domaines. Les estimateurs directs traditionnels pour des secteurs spécifiques peuvent ne pas avoir une bonne précision car la taille de l'échantillon lorsqu'on travaille sur des petits domaines est rarement grande. De là la nécessité d'utiliser les estimateurs indirects qui empruntent la force aux domaines apparentés, en particulier les estimateurs indirects basés sur les modèles explicites de liaison. Des modèles de niveau de secteur et de niveau d'unité standard et leurs extensions ont été intensivement étudiés dans la littérature pour dériver la meilleure prévision linéaire sans biais empirique (EBLUP), l'estimateur de Bayes de petits domaines empirique (EB) et l'estimateur hiérarchique de petits domaines de Bayes (HB) ainsi que les mesures de variabilité qui leur sont associées. Dans cet article, je couvre plusieurs nouveaux développements importants liés à l'estimation sur des petits domaines basée sur les modèles.

MOTS CLÉ: estimation de l'erreur quadratique moyenne; jackknife; modèles de niveau de secteur; modèles de niveau d'unité standard.

### 1. INTRODUCTION

Due to cost and other considerations, sample surveys are typically designed to provide area-specific (or direct) estimators with small sampling coefficient of variation (CV) for large areas (or domains). In fact, survey practitioners often stress that nonsampling errors, including measurement and coverage errors, contribute much more than sampling errors to total mean squared error (MSE) which is often used as a measure of quality of estimators. However, sampling errors play a dominant role in small area estimation because sample sizes in small areas are seldom large enough to provide direct estimators with acceptable quality in terms of sampling MSE (or CV). In fact, sample sizes can be zero in many small areas of interest. For example, data from the Current Population Survey (CPS) are used to estimate county (and school district) counts of poor school age children in the United States, but the CPS sample sizes are zero in many of the counties (National Research Council, 2000).

Due to difficulties with direct estimators, it is often necessary to employ indirect estimates that borrow information from related areas through explicit (or implicit) linking models, using census and administrative data associated with the small areas. Indirect estimators based on explicit linking models have received a lot of attention in recent years because of the following advantages over the traditional indirect estimators based on implicit models: (i) Explicit model-based methods make specific allowance for local variation through complex error structures in the model that link the small areas. (ii) Models can be validated from the sample data. (iii) Methods can handle complex cases such as cross-sectional and time series data, binary or count data, spatially-correlated data and multivariate data. (iv) Area-specific measures of variability associated with the estimates may be obtained, unlike overall measures commonly used with the traditional indirect estimators.

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<sup>1</sup> J.N.K. Rao (jrao@math.carleton.ca), Carleton University, School of Mathematics and Statistics, Ottawa, Canada

Basic area level and unit level models have been extensively studied in the literature to derive empirical best linear unbiased prediction (EBLUP), empirical Bayes (EB) and hierarchical Bayes (HB) small area estimators of totals (or means) and associated measures of variability. The EBLUP method is applicable for linear mixed models that cover the basic area level and unit level models. On the other hand, EB and HB methods are more generally applicable, covering generalized linear mixed models that are used to handle categorical (e.g., binary) and count data. MSE is used as a measure of variable under the EBLUP and EB approaches, while the HB approach uses the posterior variance as a measure of variability, assuming a prior distribution on the model parameters. We refer the reader to Rao (2003) for an extensive account of EBLUP, EB and HB methods for small area estimation. In this paper, I will cover several important new developments related to model-based small area estimation, including unmatched sampling and linking area level models, use of sampling weights in unit level models, jackknife methods for MSE estimation, and MSE estimation for area level models when the sampling variances are estimated.

## 2. SMALL AREA MODELS

Two types of basic small area models have been studied in the literature. In the first type, called the basic area level model, only area-specific auxiliary data  $z_i = (z_{li}, \dots, z_{pi})^T$ , related to some suitable function  $\theta_i = g(Y_i)$  of the small area total  $Y_i (i=1, \dots, m)$ , are used to develop a linking model of the form  $\theta_i = z_i^T \beta + v_i$  with  $v_i \stackrel{iid}{\sim} N(0, \sigma_v^2)$ , where  $\sigma_v^2$  is the model variance. The linking model is combined with the matching sampling model  $\hat{\theta}_i = \theta_i + e_i$ , where  $\hat{\theta}_i = g(\hat{Y}_i)$  is a direct estimator of  $\theta_i$  and  $e_i | \theta_i \stackrel{ind}{\sim} N(0, \psi_i)$  with known sampling variance  $\psi_i$ . The combined model,  $\hat{\theta}_i = z_i^T \beta + v_i + e_i$ , is a special case of the linear mixed model.

The basic area level model has at least two limitations. First, the assumption of known sampling variances,  $\psi_i$ , is restrictive, although methods based on generalized variance functions (GVF) have been proposed to produce smoothed estimates of the  $\psi_i$ 's. Secondly, the assumption  $E(e_i | \theta_i) = 0$  may not be tenable if the small area sample size,  $n_i$ , is small and  $\theta_i$  is a nonlinear function of the total  $Y_i$ , even if the director estimator  $\hat{Y}_i$  is design-unbiased for  $Y_i$ . It is more realistic to use the sampling model  $\hat{Y}_i = Y_i + f_i$  with  $E(f_i | Y_i) = 0$ , which simply says that  $\hat{Y}$  is design-unbiased for  $Y_i$ . Further, we assume that  $V(f_i | Y_i) = \sigma_i^2$ , where the sampling variance may depend on  $Y_i$ ; for example,  $\sigma_i^2 = Y_i^2 c_i^2$ , where  $c_i$  is the known coefficient of variation of  $\hat{Y}_i$  ascertained from fitting GVF's. The sampling model is now unmatched with the linking model in the sense that they cannot be combined directly to produce a linear mixed model. Various extensions of the basic area level (also called Fay-Herriot model) have been proposed to handle correlated sampling errors, spatial dependence of the model errors  $v_i$  and time-series and cross-sectional data (see Rao, 2003, Chapter 8).

In the second type, called basic unit level model, unit level auxiliary variables  $x_{ij} = (x_{1ij}, \dots, x_{pij})^T$  are related to the unit  $y$ -values  $y_{ij}$  through a nested error linear regression model  $y_{ij} = x_{ij}^T \beta + v_i + e_{ij}$ , where  $v_i \stackrel{iid}{\sim} N(0, \sigma_v^2)$  and independent of  $e_{ij} \stackrel{iid}{\sim} N(0, \sigma_e^2)$ . Various extensions of the basic unit-level model have been proposed to handle binary responses, two-stage sampling within areas, multivariate responses and others (see Rao, 2003, Chapters 8, 9 and 10).

## 3. BASIC AREA LEVEL MODEL: EB

### 3.1 Estimation of $\theta_i$

Under the basic area level model, the best estimator of  $\theta_i$  in the sense of minimum MSE is given by  $E(\theta_i | \hat{\theta}_i, \beta, \sigma_v^2)$  which depends on the model parameters  $\beta$  and  $\sigma_v^2$ . Replacing  $(\beta, \sigma_v^2)$  by suitable estimators  $(\hat{\beta}, \hat{\sigma}_v^2)$  obtained from the marginal distribution of  $\hat{\theta}_i$ 's, namely  $\hat{\theta}_i \stackrel{ind}{\sim} N(z_i^T \beta, \sigma_v^2 + \psi_i)$ , we obtain the empirical Bayes or empirical best (EB) estimator:

$$\hat{\theta}_i^{EB} = \hat{\gamma}_i \hat{\theta}_i + (1 - \hat{\gamma}_i) z_i^T \hat{\beta}, \quad (3.1)$$

where  $\hat{\gamma}_i = \hat{\sigma}_v^2 / (\hat{\sigma}_v^2 + \psi_i)$ . The form (3.1) shows that the EB estimator of  $\theta_i$  is a weighted average of the direct estimator  $\hat{\theta}_i$  and the regression synthetic estimator  $z_i^T \hat{\beta}$  with weights  $\hat{\gamma}_i$  and  $1 - \gamma_i$  respectively. The weight  $\hat{\gamma}_i$  is a measure of between area variability relative to total variability associated with area  $i$ . The estimator  $\hat{\theta}_i^{EB}$  is unbiased for  $\theta_i$  under the combined model, i.e.,  $E(\hat{\theta}_i^{EB} - \theta_i) = 0$ , and design-consistent as the sampling variance  $\psi_i$  goes to zero, provided the direct estimator is design-consistent. However, the resulting estimator  $g^{-1}(\hat{\theta}_i^{EB})$  of  $Y_i$  is biased if  $g(\cdot)$  is nonlinear. Note that  $g^{-1}(\hat{\theta}_i^{EB})$  is not equal to the EB estimator  $\hat{Y}_i^{EB}$  obtained by evaluating  $E[g^{-1}(\theta_i) | \hat{\theta}_i, \beta, \sigma_v^2]$  at  $\hat{\beta}$  and  $\hat{\sigma}_v^2$ .

Under normality assumption, maximum likelihood (ML) or residual maximum likelihood (REML) method may be used to estimate  $\beta$  and  $\sigma_v^2$  from the marginal distribution  $\hat{\theta}_i \stackrel{iid}{\sim} N(z_i^T \beta, \sigma_v^2 + \psi_i)$ . Alternatively,  $\sigma_v^2$  may be estimated by a simple method of moments (Prasad and Rao, 1990) or by solving the following moment equation iteratively for  $\sigma_v^2$  (Fay and Herriot, 1979):

$$a(\sigma_v^2) = \sum_{i=1}^m (\hat{\theta}_i - z_i^T \tilde{\beta}(\sigma_v^2))^2 / (\sigma_v^2 + \psi_i) = m - p, \quad (3.2)$$

where  $\tilde{\beta}(\sigma_v^2)$  is the weighted least squares estimator of  $\beta$  for given  $\sigma_v^2$ . The resulting estimators  $\hat{\sigma}_v^2$  and  $\hat{\beta} = \tilde{\beta}(\hat{\sigma}_v^2)$  lead to the EBLUP estimator of  $\theta_i$  from (3.1). The EBLUP estimator does not depend on normality.

### 3.2 MSE Estimation

Methods of estimating  $MSE(\hat{\theta}_i^{EB})$  that account for the variability of  $\hat{\beta}$  and  $\hat{\sigma}_v^2$  have been studied extensively in the literature, where  $MSE(\hat{\theta}_i^{EB}) = E(\hat{\theta}_i^{EB} - \theta_i)^2$  and the expectation is with respect to the combined model (see Rao, 2003, Chapter 7). An accurate approximation to  $MSE(\hat{\theta}_i^{EB})$  under normality is given by

$$MSE(\hat{\theta}_i^{EB}) \approx g_{1i}(\sigma_v^2) + g_{2i}(\sigma_v^2) + g_{3i}(\sigma_v^2) \quad (3.3)$$

where the leading term  $g_{1i}(\sigma_v^2) = \gamma_i \psi_i$  with  $\gamma_i = \sigma_v^2 / (\sigma_v^2 + \psi_i)$  is the contribution to MSE assuming  $\beta$  and  $\sigma_v^2$  are known,

$$g_{2i}(\sigma_v^2) = (1 - \gamma_i)^2 z_i^T \left[ \sum_{i=1}^m z_i z_i^T / (\sigma_v^2 + \psi_i) \right]^{-1} z_i \quad (3.4)$$

accounts for the variability of  $\hat{\beta}$  and the term

$$g_{3i}(\sigma_v^2) = \left[ \psi_i^2 / (\sigma_v^2 + \psi_i)^4 \right] E(\hat{\theta}_i - z_i^T \beta)^2 h(\sigma_v^2) \quad (3.5)$$

$$= \left[ \psi_i^2 / (\sigma_v^2 + \psi_i)^3 \right] h(\sigma_v^2) \quad (3.6)$$

accounts for the variability of  $\hat{\sigma}_v^2$ , where  $h(\sigma_v^2)$  is the asymptotic variance of  $\hat{\sigma}_v^2$  for large  $m$ . Neglected terms in the approximation (3.3) are of lower order than  $m^{-1}$ , and the approximation is valid for the Prasad-Rao (PR), Fay-Herriot (FH), ML and REML methods of estimating  $\sigma_v^2$ .

Comparing the leading term  $\gamma_i \psi_i$  of (3.3) with  $\psi_i$ , the MSE of the direct estimator  $\hat{\theta}_i$ , it is clear that the EB estimator  $\hat{\theta}_i^{EB}$  leads to large gain in efficiency when  $\gamma_i$  is small, i.e., when  $\sigma_v^2$ , the variability of the model errors  $v_i$ , is small relative to the total variability,  $\sigma_v^2 + \psi_i$ . Note that  $\psi_i$  is also the design variance of  $\hat{\theta}_i$ .

Turning to MSE estimation, an estimator correct to the same order approximation as (3.3) is given by

$$mse(\hat{\theta}_i^{EB}) \approx g_{1i}(\hat{\sigma}_v^2) + g_{2i}(\hat{\sigma}_v^2) + 2g_{3i}(\hat{\sigma}_v^2). \quad (3.7)$$

The estimator (3.7) is approximately unbiased for  $MSE(\hat{\theta}_i^{EB})$  in the sense that its bias is of lower order than  $m^{-1}$ , provided  $\hat{\sigma}_v^2$  is based on REML or PR. For ML and FH methods of estimating  $\sigma_v^2$ , an extra term  $g_{0i}(\hat{\sigma}_v^2)$  is added to (3.7). This extra term for ML is positive (Datta and Lahiri, 2000). Therefore, ignoring this term and using (3.7) with ML estimator  $\hat{\sigma}_v^2$  would lead to underestimation of MSE. On the other hand, the extra term for FH is negative (Datta, Rao and Smith, 2002). Therefore, ignoring this term and using (3.7) with FH estimator  $\hat{\sigma}_v^2$  would lead to overestimation of MSE.

A criticism of the MSE estimator (3.7) and its modification for ML and FH is that it is not area-specific in the sense that it does not explicitly depend on  $\hat{\theta}_i$  although the area-specific auxiliary data  $z_i$  is involved in the  $g_{3i}(\hat{\sigma}_v^2)$ -term. Rao (2000) used the expression (3.5) for  $g_{3i}(\sigma_v^2)$  to get an alternative area-specific estimator of  $g_{3i}(\sigma_v^2)$ :

$$\tilde{g}_{3i}(\hat{\sigma}_v^2, \hat{\theta}_i) = \left[ \psi_i^2 / (\sigma_v^2 + \psi_i)^4 \right] (\hat{\theta}_i - z_i^T \hat{\beta})^2 h(\hat{\sigma}_v^2). \quad (3.8)$$

Using (3.8), we get two different area-specific MSE estimators for REML or PR:

$$mse_1(\hat{\theta}_i^{EB}) \approx g_{1i}(\hat{\theta}_v^2) + g_{2i}(\hat{\sigma}_v^2) + g_{3i}(\hat{\sigma}_v^2) + \tilde{g}_{3i}(\hat{\sigma}_v^2, \hat{\theta}_i) \quad (3.9)$$

and

$$mse_2(\hat{\theta}_i^{EB}) \approx g_{1i}(\hat{\theta}_v^2) + g_{2i}(\hat{\sigma}_v^2) + 2\tilde{g}_{3i}(\hat{\sigma}_v^2, \hat{\theta}_i). \quad (3.10)$$

The term  $\tilde{g}_{3i}(\hat{\sigma}_v^2, \hat{\theta}_i)$  is less stable than  $g_{3i}(\hat{\sigma}_v^2)$  but it is of lower order than the leading term  $g_{1i}(\hat{\sigma}_v^2)$  in (3.9) and (3.10). As a result, the coefficient of variation (CV) of  $mse_1(\hat{\theta}_i^{EB})$  should be comparable to the CV of  $mse(\hat{\theta}_i^{EB})$ , at least for moderate to large  $m$ . Fuller (1989) estimated the conditional MSE of  $\hat{\theta}_i^{EB}$  given the  $i$ -th area direct estimator  $\hat{\theta}_i$ . His area-specific MSE estimator is closely related to the unconditional MSE estimator (3.9). Butar and Lahiri (1997) obtained an area-specific MSE estimator by correcting the bias of Laird and Louis' (1987) estimator of variability based on the parametric bootstrap method. This bias-corrected MSE estimator is identical to (3.9) which is obtained in a straightforward manner from the formula (3.5) for  $g_{3i}(\sigma_v^2)$ .

It is more appealing to survey practitioners to consider the estimation of sampling MSE of  $\hat{\theta}_i^{EB}$ , i.e.,  $MSE_p(\hat{\theta}_i^{EB}) = E_p(\hat{\theta}_i^{EB} - \theta_i)^2$ , where the expectation  $E_p$  is with respect to the sampling design  $p(\cdot)$ , i.e., the distribution of sampling errors given the  $\theta_i$ 's. Rivest and Belmonte (2000) derived a design-unbiased estimator of  $MSE(\hat{\theta}_i^{EB})$  using the PR estimator of  $\sigma_v^2$ . The leading term of this MSE estimator is area-specific, i.e., depends on  $\hat{\theta}_i$ , unlike the leading term  $g_{1i}(\hat{\sigma}_v^2)$  of the model-based MSE estimator,  $mse(\hat{\theta}_i^{EB})$ . However, it is highly unstable relative to  $mse(\hat{\theta}_i^{EB})$  unless more weight is attached to the direct estimator  $\hat{\theta}_i$ , i.e.,  $1 - \hat{\gamma}_i$  is small.

### 3.3 Unknown Sampling Variances $\psi_i$

In sections 4.1 and 4.2 we assumed that the sampling variances,  $\psi_i$ , are known, but this is a restrictive assumption. Wang and Fuller (2003) and Rivest and Vandal (2002) studied the effect of estimating  $\psi_i$  on the MSE of the EB estimator (4.1) with  $\hat{\gamma}_i$  replaced by  $\hat{\sigma}_v^2 / (\hat{\sigma}_v^2 + \hat{\psi}_i)$ , where  $\hat{\psi}_i$  is an estimator of  $\psi_i$ . For example, suppose that we have a random sample  $y_{ij} \stackrel{iid}{\sim} N(\theta_i, \sigma^2)$ ,  $j=1, \dots, n_i (\geq 2)$  from the  $i$ -th area and  $\hat{\theta}_i = \bar{y}_i$ , the sample mean. In this case  $\hat{\psi}_i = s_i^2 / n_i$  is design-unbiased for  $\psi_i$ , where  $s_i^2$  is the sample variance. Further,  $\bar{y}_i$  and  $\hat{\psi}_i$  are independently distributed with  $\hat{\psi}_i \approx N[\psi_i, \delta_i = 2\psi_i^2 / (n_i - 1)]$ . Under this set-up, Rivest and Vandal (2002) obtained an appropriate MSE estimator by adding the term  $2\hat{\delta}_i \hat{\sigma}_v^4 / (\hat{\psi}_i + \hat{\sigma}_v^2)^3$  to (3.8) to account for the estimation of  $\psi_i$ , where  $\hat{\delta}_i = 2\hat{\psi}_i^2 / (n_i - 1)$ . If the sample sizes  $n_i$  are small, then (3.8) can underestimate the MSE quite severely, unlike the Rivest-Vandal MSE estimator. If  $\hat{\psi}_i$  is a smoothed estimator of  $\psi_i$  based on GVF model fitting, the contribution from the extra term is of the same order,  $O(m^{-1})$ , as the  $g_{3i}$ -term.

#### 4. JACKKNIFE ESTIMATION OF MSE

Jiang, Lahiri and Wan (2002) proposed a jackknife method of estimating MSE of EB estimators that is applicable to generalized linear mixed models with block diagonal covariance structures, where the blocks correspond to small areas. This method also leads to approximately unbiased estimators of MSE of EB estimators. For example, consider the case of binary responses  $y_{ij} \stackrel{iid}{\sim} \text{Bernoulli}(p_i)$ ,  $j=1, \dots, n_i$  and  $\log\{p_i / (1 - p_i)\} = z_i^T \beta + v_i$ ,  $i=1, \dots, m$ , where  $z_i$  is the vector of area-specific covariates,  $v_i \stackrel{iid}{\sim} N(0, \sigma_v^2)$  and  $p_i$  is the  $i$ -th area proportion. The minimum MSE (or Bayes) estimator of  $p_i$  is given by  $\hat{p}_i^B = E(p_i | y_i, \beta, \sigma_v^2) = k(y_i, \beta, \sigma_v^2)$ , where  $y_i = \sum_j y_{ij}$ . The EB estimator of  $p_i$  is  $\hat{p}_i^{EB} = k(y_i, \hat{\beta}, \hat{\sigma}_v^2)$ , where  $\hat{\beta}$  and  $\hat{\sigma}_v^2$  are suitable estimators of  $\beta$  and  $\sigma_v^2$  obtained from the marginal distribution of  $y_i$ 's.

The jackknife method makes use of the following orthogonal decomposition of  $MSE(\hat{p}_i^{EB})$ :

$$MSE(\hat{p}_i^{EB}) = E(\hat{p}_i^B - p_i)^2 + E(\hat{p}_i^{EB} - \hat{p}_i^B)^2. \quad (4.1)$$

Based on the decomposition (4.1), Jiang et al. (2002) proposed the following jackknife steps to estimate  $MSE(\hat{p}_i^{EB})$ :

- (1) Calculate  $\hat{\beta}(\ell)$  and  $\hat{\sigma}_v^2(\ell)$  deleting the  $P$ -th area data  $(y_i, x_i)$ . Let  $\hat{p}_i^{EB}(\ell) = k\{y_i, \hat{\beta}(\ell), \hat{\sigma}_v^2(\ell)\}$  be the EB estimator of  $p_i$  based on  $\hat{\beta}(\ell)$  and  $\hat{\sigma}_v^2(\ell)$ . Note that  $y_i$  remains unchanged.
- (2) Calculate the jackknife estimator of the last term in (4.1) as

$$\hat{M}_{2i} = \frac{m-1}{m} \sum_{\ell=1}^m [\hat{p}_i^{EB}(\ell) - \hat{p}_i^{EB}]^2. \quad (4.2)$$

- (3) The first term  $E(\hat{p}_i^B - p_i)^2$  may be written as  $E[\tilde{g}_{1i}(y_i, \beta, \sigma_v^2)] =: g_{1i}(\beta, \sigma_v^2)$ , where  $\tilde{g}_{1i}(y_i, \beta, \sigma_v^2) = V(p_i | y_i, \beta, \sigma_v^2)$  is the posterior variance of  $p_i$  given  $y_i$  and  $(\beta, \sigma_v^2)$ . Adjust the bias of  $g_{1i}(\hat{\beta}, \hat{\sigma}_v^2)$  (as an estimator of  $g_{1i}(\beta, \sigma_v^2)$ ) using the jackknife bias reduction method. The bias-adjusted estimator is given by

$$\hat{M}_{1i} = g_{1i}(\hat{\beta}, \hat{\sigma}_v^2) - \frac{m-1}{m} \sum_{\ell=1}^m [g_{1i}(\hat{\beta}(\ell), \hat{\sigma}_v^2(\ell)) - g_{1i}(\hat{\beta}, \hat{\sigma}_v^2)]. \quad (4.3)$$

Note that the leading term  $g_{1i}(\hat{\beta}, \hat{\sigma}_v^2)$  is not area-specific in the sense of not depending on  $y_i$ .

- (4) Calculate the jackknife estimator of MSE as

$$mse_J(\hat{p}_i^{EB}) = \hat{M}_{1i} + \hat{M}_{2i}. \quad (4.4)$$

Booth and Hobert (1998) argued that for non-normal responses the MSE estimator should be area-specific because the posterior variance of  $p_i$  given  $(\beta, \sigma_v^2)$  depends on  $y_i$ , unlike the case of linear mixed models. They proposed the conditional MSE, given the  $i$ -th area data  $(y_i, z_i)$ , as the relevant measure of variability, and estimated the conditional MSE. Rao (2003, Chapter 9) addressed this criticism by simply modifying the bias-adjusted estimator  $\hat{M}_{1i}$ . Instead of evaluating the expectation of  $\tilde{g}_{1i}(y_i, \beta, \sigma_v^2)$  with respect to the marginal distribution of  $y_i$  (using numerical integration), he proposed to adjust the bias of  $\tilde{g}_{1i}(y_i, \hat{\beta}, \hat{\sigma}_v^2)$  as an estimator of  $g_{1i}(y_i, \beta, \sigma_v^2)$ . This leads to

$$\tilde{M}_{1i}(y_i) = \tilde{g}_{1i}(y_i, \hat{\beta}, \hat{\sigma}_v^2) - \frac{m-1}{m} \sum_{\ell=1}^m [\tilde{g}_{1i}(y_i, \hat{\beta}(\ell), \hat{\sigma}_v^2(\ell)) - \tilde{g}_{1i}(y_i, \hat{\beta}, \hat{\sigma}_v^2)] \quad (4.5)$$

which is area-specific including the leading term  $\tilde{g}_{1i}(y_i, \hat{\beta}, \hat{\sigma}_v^2)$ . The modified jackknife estimator of MSE is given by

$$mse_J^*(\hat{p}_i^{EB}) = \tilde{M}_{1i}(y_i) + \hat{M}_{2i}. \quad (4.6)$$

Note that (4.6) is not only area-specific but also computationally simpler than (4.4) because it avoids the evaluation of the expectation of  $\tilde{g}_{1i}(y_i, \beta, \sigma_v^2)$  with respect to the marginal distribution of  $y_i$ . For the case of a linear mixed model,  $\tilde{M}_{1i}(y_i) = \hat{M}_{1i}$  and hence (4.6) is identical to (4.4).

## 5. BASIC UNIT LEVEL MODEL: PSEUDO-EB

We now turn to the basic unit level model,  $y_{ij} = x_{ij}^T \beta + v_i + e_{ij}$ , and assume that the model holds for the sample, i.e., no sample selection bias within areas. If the areas are also sampled, then we assume the absence of selection bias for sampled areas as well. The mean for the  $i$ -th area,  $\bar{Y}_i$ , may be approximated as  $\mu_i = \bar{X}_i^T \beta + v_i$ , assuming that the number of population units in the  $i$ -th area,  $N_i$ , is large, where  $\bar{X}_i$  is the population mean of  $x$  for the  $i$ -th area. Assuming  $v_i \stackrel{iid}{\sim} N(0, \sigma_v^2)$  and independent of  $e_{ij} \stackrel{iid}{\sim} N(0, \sigma_e^2)$  and estimating the model parameters  $\beta$  and  $\sigma_v^2$  from the marginal distribution of the sampled  $y_{ij}$ 's, we get the EB estimator of  $\mu_i$  as

$$\hat{\mu}_i^{EB} = \hat{\gamma}_i \left[ \bar{y}_i + (\bar{X}_i - \bar{x}_i)^T \hat{\beta} \right] + (1 - \hat{\gamma}_i) \bar{X}_i^T \hat{\beta}, \quad (5.1)$$

where  $\hat{\gamma}_i = \hat{\sigma}_v^2 / (\hat{\sigma}_v^2 + \hat{\sigma}_e^2 / n_i)$ ,  $(\bar{y}_i, \bar{x}_i)$  are the  $i$ -th area sample means and  $(\hat{\beta}, \hat{\sigma}_v^2)$  are the estimators of  $(\beta, \sigma_v^2)$ ; see Battese, Harter and Fuller (1988). This estimator is also the EBLUP estimator without normality assumption, provided  $\beta$  and  $\sigma_v^2$  are estimated by a moments method such as the method of fitting-of-constants. As  $n_i \rightarrow \infty$ , the EB estimator converges to the ‘‘survey regression’’ estimator  $\bar{y}_i + (\bar{X}_i - \bar{x}_i)^T \hat{\beta}$ , while it converges to the ‘‘synthetic regression’’ estimator  $\bar{X}_i^T \hat{\beta}$  as  $n_i$  decreases. We refer to Rao (2003), Chapter 7 for a detailed discussion of EBLUP estimation and MSE estimation. A drawback of (5.1) is that it is purely model-based and does not take account of the survey weights  $w_{ij}$ . As a result, it is not design-consistent as  $n_i$  increases, unless the sampling design is self-weighting within areas, i.e.,  $w_{ij} = w_i$ . On the other hand, the EBLUP estimator under the area level model is design-consistent. It is desirable to ensure design-consistency for the unit level models as well because  $n_i$  could be moderately large for some of the areas under consideration. Also, it is desirable to ensure that the estimators of area totals automatically add up to the direct survey regression estimator of the large area total. You and Rao (2002a) developed a pseudo-EBLUP estimator of  $\mu_i$  that satisfies both the desirable properties.

We have assumed that the random small area effects  $v_i$  are normally distributed in the basic unit level model  $y_{ij} = x_{ij}^T \beta + v_i + e_{ij}$ . The MSE estimator of the purely model-based estimator  $\hat{\mu}_i^{EB}$  derived under normality of the  $v_i$ 's is not robust to deviations from normality, unlike the case of basic area level model studied in Section 3.2. It would be worthwhile to study EB inference under semi-nonparametric (SNP) representations of the density of  $v_i$ . Zhang and Davidian (2001) approximated the density of  $v_i$  by a SNP representation which includes normality as a special case, and it provides flexibility in capturing nonnormality through a user-chosen tuning parameter. Maiti (2001) used a finite mixture of normal distributions for the distribution of  $v_i$ , and developed hierarchical Bayes (HB) estimates of small area means, assuming a prior distribution on the model parameters. EB estimation of small area means and associated MSE estimation under broad classes of densities of  $v_i$ , such as the above, would be practically useful.

## 6. HIERARCHICAL BAYES (HB) APPROACH

We illustrate the HB approach using the basic area level model of Section 2. Under this approach, a prior distribution on the model parameters  $\delta = (\beta^T, \sigma_v^2)$  is specified, and inferences are then based on the posterior distribution,  $f(\theta_i | \hat{\theta})$ , of  $\theta_i$  given the data  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_m)^T$ . In particular,  $\theta_i$  is estimated by its posterior mean  $E(\theta_i | \hat{\theta})$ , called the HB estimator  $\hat{\theta}_i^{HB}$ . The variability of  $\hat{\theta}_i^{HB}$  is measured by the posterior variance  $V(\theta_i | \hat{\theta})$ . The calculation of  $\hat{\theta}_i^{HB}$  and  $V(\theta_i | \hat{\theta})$  involves integrations with respect to the posterior distribution of  $\beta, \sigma_v^2, f(\beta, \sigma_v^2 | \hat{\theta})$ . However, Monte Carlo Markov chain (MCMC) methods can be used to generate  $J$  simulated samples  $\{\theta_1^{(j)}, \dots, \theta_m^{(j)}; j = 1, \dots, J\}$  directly from the joint posterior  $f(\theta | \hat{\theta})$ , where  $\theta = (\theta_1, \dots, \theta_m)^T$  and  $J$  is sufficiently large. Using the simulated samples, we approximate  $\hat{\theta}_i^{HB}$  by the mean  $\theta_i^{(c)} = J^{-1} \sum_j \theta_i^{(j)}$  and  $V(\theta_i | \hat{\theta})$  by the variance  $J^{-1} \sum_j (\theta_i^{(j)} - \theta_i^{(c)})^2$  of the simulated samples  $\theta_i^{(j)}$ . The HB estimator of the total  $Y_i$  is approximated by the mean  $Y_i^{(c)} = J^{-1} \sum_j Y_i^{(j)}$  and the posterior variance of  $Y_i, V(Y_i | \hat{\theta})$  by the variance  $J^{-1} \sum_j (Y_i^{(j)} - Y_i^{(c)})^2$ , where  $Y_i^{(j)} = g^{-1}(\theta_i^{(j)})$ .

An advantage of the HB approach is that it is straightforward, the inferences are “exact” unlike the EB (or EBLUP) approach, and it can handle complex small area models using MCMC methods, but it requires the specification of a prior  $f(\beta, \sigma_v^2)$  on the model parameters. It would be desirable to select a “matching” prior  $f(\beta, \sigma_v^2) \propto f(\sigma_v^2)$  that leads to well-calibrated inferences. In particular, the posterior variance should be approximately unbiased for  $MSE(\hat{\theta}_i^{HB})$ , i.e.,  $E[V(\theta_i | \hat{\theta})] - MSE(\theta_i^{HB}) = o(m^{-1})$ ; asymptotically,  $\hat{\theta}_i^{HB} \approx \hat{\theta}_i^{EB}$ . This will provide a frequentist justification for the posterior variance as a measure of variability Datta, Rao and Smith (2002) showed that the matching prior is given by

$$f_i(\sigma_v^2) \propto (\sigma_v^2 + \psi_i)^2 \sum_{\ell=1}^m (\sigma_v^2 + \psi_\ell)^{-2}. \quad (6.1)$$

This prior depends collectively on the sampling variances,  $\psi_\ell$ , for all the areas  $P$  as well as on the area-specific sampling variance  $\psi_i$ . For the balances case,  $\psi_i = \psi$ , the matching prior reduces to the “flat” prior  $f(\sigma_v^2) \propto 1$ . Note that the prior (6.1) on the common parameter  $\sigma_v^2$  is designed for inference on the  $i$ -th area so that its dependence on  $\psi_i$  may not be problematic.

A disadvantage of the EB estimator  $\hat{\theta}_i^{EB}$  is that the weight  $\hat{\gamma}_i$  attached to the direct estimator takes zero value when  $\hat{\sigma}_v^2 = 0$ , in which case it reduces to the regression synthetic estimators  $z_i^T \hat{\beta}$ . Thus, all the direct estimator  $\hat{\theta}_i$  receive zero weight even when the sample sizes in some areas are not small. This difficulty was encountered in using a state model to produce EB state estimates of poor school-age children in the United States (National Research Council, 2000). The HB approach avoids this difficulty by producing positive weights in all cases. Bell (1999) applied the HB approach to the

state model using the prior  $f(\beta, \sigma_v^2) = f(\beta)f(\sigma_v^2)$  with  $f(\beta) \propto 1$  and  $f(\sigma_v^2) \propto 1$ , and obtained positive weights in all cases. But it is not clear if his method leads to well-calibrated inferences since the matching prior (6.1) is different from the flat prior, especially when the  $\psi_i$ -values vary significantly, as in the case of Bell (1999) with  $\max(\psi_i) / \min(\psi_i)$  as large as 20.

You and Rao (2002b) used the HB approach to handle the case of unmatched sampling and linking area level models (Section 2) and applied it to Canadian census undercount estimation. Singh, Folsom and Vaish (2003) studied the basic unit level model,  $y_{ij} = x_{ij}^T \beta + v_i + e_{ij}$  with  $v_i \stackrel{iid}{\sim} N(0, \sigma_v^2)$  independent of  $e_{ij} \stackrel{iid}{\sim} N(0, \sigma_e^2)$ , for the population, and allowed sample selection bias within small areas. They used methods based on survey-weighted estimating functions (EF) to account for the sample selection bias. They also extended the method to generalized linear mixed model such as the logistic mixed model  $y_{ij} | p_{ij} \stackrel{ind}{\sim} \text{Bernoulli}(p_{ij})$  and  $\text{logit}(p_{ij}) = \log\{p_{ij} / (1 - p_{ij})\} = x_{ij}^T \beta + v_i$ ; see Rao (2003, pp. 253-254) for a brief account of the EF method. In situations where a sample of areas is selected, the random effects  $v_i$  are assumed to be free of sample selection bias.

The HB approach is powerful and attractive, but caution should be exercised when using MCMC methods. For example, MCMC algorithms could lead to seemingly reasonable inferences about a non-existent posterior distribution. This happens when the posterior distribution is improper and yet all the Gibbs conditional distributions, used in generating the MCMC simulated samples, are proper (Hobert and Casella, 1996). Another difficulty with MCMC is that the convergence diagnostic tools can fail to detect the sort of convergence failures they were designed to identify (Cowles and Carlin, 1996). We refer the reader to Rao (2003), Section 10.2.4 for a discussion of practical issues associated with MCMC.

MCMC methods are also extensively used for model determination which plays a vital role in developing model-based small area estimates. In particular, methods based on Bayes factors, posterior predictive densities and cross-validation predictive densities are employed for model determination; see Rao (2003), Section 10.2.6. The criterion of posterior predictive probability is often used to check the overall fit of a proposed model. Sinharay and Stern (2003) conducted a simulation study to investigate the effectiveness of this criterion for model checking, using the basic area level model with no covariates. Their study indicates that it is difficult to detect nonnormality of the random effects  $v_i$  using this criterion, unless the extent of violation is huge. Therefore, caution should be exercised in using HB criteria for model determination.

## 7. SOME PRACTICAL CONSIDERATIONS

In this section, we provide brief remarks on practical issues related to small area estimation.

### (i) Design issues

It is important to consider design issues that have an impact on small area estimation. A proper resolution of design issues could lead to enhancement in the reliability of direct as well as indirect estimates for both planned and unplanned domains (areas). The following measures at the design stage might be useful in minimizing the need for indirect estimators, at least for some planned domains: Use of list frames to replace clusters wherever possible, use of many small strata from which samples are drawn, compromise sample allocations to satisfy reliability requirements at a small area level as well as large area level, integration of surveys through harmonizing questions across surveys of the same population, use of multiple frame surveys, use of “rolling samples” as a method of cumulating data over time. We refer the reader to Rao (2003), Chapter 2, Singh et al. (1994) and Marker (2001) for further details.

Despite the above preventive measures at the design stage, indirect estimates will be needed in practice because it is not possible to anticipate and plan for all possible areas (or domains) and uses of survey data: “the client will always require more than is specified at the design stage” (Fuller, 1999, p. 344).

### (ii) Model selection and validation



Methodological developments and applications of model-based estimation are impressive, but caution should be exercised because of the model assumptions. Good auxiliary information related to the variables of interest plays a vital role in determining suitable linking models. Therefore, more attention should be given to the compilation of auxiliary variables that are good predictors of study variables. Subject matter specialists or end users should have influence on the choice of models, particularly on the choice of auxiliary variables. However, model diagnostics should be used to find suitable model(s) that fit the data well. Such model diagnostics include residual analysis to detect departures from assumed models, selection of auxiliary variables and case-deletion diagnostics to detect influential observations. We refer the reader to Rao (2003), Chapter 6 for some methods of model validation in the frequentist framework.

Hierarchical Bayes (HB) approach has become very popular in recent years due to its ability to handle complex models using MCMC methods. However, caution should be exercised in the choice of improper priors on model parameters, as noted in Section 6. Limitations of MCMC methods, such as shortcomings of available convergence diagnostics, should also be noted. Carlin and Louis (2000) made an important observation on the dangers of “plug and play” implementation via MCMC: “Worse, the sheer power of MCMC methods has led to the temptation to fit models larger than the data can readily support without a strongly informative prior structure - now something of a rarity in applied Bayesian work”.

HB methods for model validation via MCMC have been extensively developed, but the effectiveness of some criteria for model checking is questionable as noted in Section 6. Further work on effective HB methods for model checking is needed.

### **(iii) Area level vs. unit level models**

Area level models have wider scope than unit level models because area level auxiliary information is more readily available than unit level auxiliary data. Also, design weights are incorporated by modelling design-weighted direct estimators, and the resulting EB or HB estimators are design consistent. But the assumption of known sampling variances,  $\psi_i$ , is quite restrictive. Smoothed estimates of  $\psi_i$ 's based on GVF model fitting can also cause difficulties in MSE estimation, as noted in Section 3.3. We need more work on obtaining good approximations to the sampling variances as well as methods that incorporate the variability associated with estimated sampling variances in MSE estimation. This task becomes more difficult when using multivariate or time series area level models because sampling covariances are also needed.

Recent work on incorporating survey weights into model-based estimation, via pseudo-EB or pseudo-HB, is promising; in particular, the self-benchmarking property noted in Section 5. But the assumption that the sample selection bias is absent may not be true for some applications. The estimating functions (EF) approach of Singh, Folsom and Vaish (2003), mentioned in Section 6, allows sample selection bias within sampled areas but it assumes that the random effects  $v_i$  are free from sample selection bias in situations where a sample of areas is selected. Methods for handling the latter case are needed. Moreover, their method assumes known sampling variances, as in the area level model, and this assumption may be restrictive (see Rao, 2003, Section 10.5.4).

### **(iv) “Triple-goal” estimation**

We focussed on model-based estimation of small area totals or means, but such estimates may not be suitable if the main objective is to produce an ensemble of parameter estimates whose distribution is in some sense close enough to the distribution of area-specific parameters. For example, we may be interested in ranking areas or identifying areas that fall below or above some prespecified level. Shen and Louis (1998) proposed “triple-goal” estimators that can produce good ranks, a good histogram and good area-specific estimators, assuming simple linking models. It would be useful to extend their methods to handle more complex models that are suitable for small area estimation.

### **(v) Nonsampling errors**

We have assumed the absence of measurement errors in the responses and/or the covariates as well as nonresponse. But nonsampling errors can have a substantial effect on small area estimation, and it would be useful to develop suitable designs as well as methods of estimation that can account for nonsampling errors. Nandram and Choi (2002) used HB nonresponse models for binary data, and applied the theory to data from the U.S. National Crime Survey to estimate small area proportions. Measurement errors in the responses, even under an additivity assumption, can lead to biased estimators

of quantiles and histograms. In the context of direct estimation, Fuller (1995) proposed methods at the design stage that can lead to bias-adjusted estimators of quantiles and histograms.

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